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Study of Hénon's Orbit Transfer Problem Using the Lambert Algorithm

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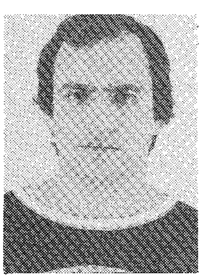
The problem of transfer orbits from one body back to the same body (the moon or a planet) is formulated as a Lambert problem and solved by Gooding's Lambert routines. Elliptic and circular orbits are considered for the moon or a planet and any type of orbit (elliptic, parabolic, or hyperbolic) is considered for the spacecraft. The solutions are plotted in terms of the true anomaly (instead of the eccentric anomaly) for several cases. The use of the true anomaly simplifies the solutions in several ways. The problem of transfers from this body to the corresponding L_4 and L_5 points is also solved. Next, the same problem is studied in terms of the ΔV and the time required for the transfer. Among all of the possible transfer orbits, a small family with almost zero ΔV was studied in detail. A transfer from the moon to the corresponding Lagrangian equilibrium points L_4 or L_5 is shown, as an example of a practical application of this theory.

Nomenclature

A	= specify if M_3 passes by periapse (1) or apoapse (0) at $t = 0$
a	= semimajor axis
e	= eccentricity
L	= "short way" (1) or "long way" (0) transfer
L_4, L_5	= Lagrangian points
M_1	= primary body
M_2	= secondary body
M_3	= third body (spacecraft)
m	= number of revolutions during transfer
n	= mean angular velocity
PA	= periapse in a positive (1) or negative (0) abscissa
P, Q	= extreme points of a transfer
R	= radius vector
S	= transfer is direct (1) or retrograde (0)
S_3	= specify if M_2 passes by periapse (+1) or apoapse (−1) at $t = 0$
t	= time
V	= velocity
Δt	= transfer time
ΔV	= velocity increment in canonical units
ζ	= hyperbolic eccentric anomaly
η	= eccentric anomaly
$\underline{\eta}$	= redefined eccentric anomaly
$\bar{\mu}$	= mass of M_2 in canonical units
ν	= true anomaly
$\underline{\nu}$	= redefined true anomaly
τ	= half of the transfer time
ϕ	= angle between P and Q

Subscripts

1	= initial point (P)
2	= final point (Q)
R	= use of real units
r	= radial
t	= transverse



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Introduction

THE problem of transfer orbits from one body back to the same body (the moon or a planet) has been under investigation for a long time. Hénon[1] originally developed a timing condition in the eccentric anomaly for orbits that allows a spacecraft to leave the massless body M_2 (the moon or a planet in our case), go to an orbit around the other primary M_1 (the Earth or the sun in our case), and meet M_2 again after a certain time. This was treated as the problem of consecutive collision orbits in the restricted three-body problem. Several authors then worked on improvements of this problem. Hitzl[2] and Hitzl and Hénon[3,4] studied stability and critical orbits. Perko[5] derived a proof of existence and a timing condition for what was shown later to be a special case of Hénon's work. Results for the perturbed case $\mu > 0$ (where M_2 has non-negligible mass and perturbs the orbit of M_3 around M_1) also appeared in the literature. Some examples are the papers published by Gomez and Ollé[6,7] and Bruno.[8] Howell[9] and Howell and Marsh[10] extended Hénon's results for the case where the orbit of M_2 is elliptic.

In this paper this problem is formulated as that of an orbit transfer, which can be solved with Gooding's implementation of the Lambert problem.[11] The solution is given in terms of the true anomaly instead of the eccentric anomaly as has always been done by previous authors. This proves to be an interesting approach. Both cases, with the target body (moon or planet) in a circular orbit or in an elliptic orbit, are considered in the present paper. All possible types of orbits for the spacecraft M_3 are considered: elliptic, parabolic, and hyperbolic. At the same time a new problem has been solved: the transfer of the spacecraft from M_2 to the corresponding L_4 or L_5 points, which are in the same circular orbit as M_2, either 60 deg ahead of it or 60 deg behind it. The implementation developed here is generic with respect to this angle and allows us to study a transfer from M_2 to any point in the same circular orbit (not only 60 deg ahead or behind it). Next, these transfer orbits are studied in terms of the ΔV and the time required for the transfer. The various ΔV are plotted against the transfer time for several cases, and a family of transfer orbits with very small ΔV (on the order of 0.001 in canonical units, a system of units where the gravitational constant of M_1, the angular velocity of M_2, and the distance between M_1 and M_2 are all unity) is shown to exist in almost all cases studied. These orbits are studied in detail. They consist of a family of slightly different orbits (when compared with the orbit of M_2) that meet all of the requirements to provide the transfer desired. It is important to emphasize that only the limiting case $\mu = 0$ and two-impulse transfers are considered in this paper. The case $\mu \neq 0$ requires a completely different approach.

Formulation of the Problem

Let M_1 and M_2 be the two primaries with masses $(1 - \mu)$ and μ, respectively; M_2 is in a circular (in the original version of the problem studied by Hénon) or elliptic (in the extension made by Howell[9] and Howell and Marsh[10]) orbit around M_1. The massless spacecraft M_3 leaves M_2 from a point $P(t = -\tau)$, follows an orbit around M_1, and meets again with M_2 at a point $Q(t = \tau)$. Since only the limiting case $\mu = 0$ is considered, the three-body problem is reduced to the two-body problem, and the basic equations of the Kepler problem apply. It is also assumed that all three bodies involved are point masses and there are no perturbations from other bodies.

The solution to be found is the coordinate of the point P as a function of the transfer time. The solution is not unique, and a graph including many solutions was published by Hénon.[1] He plotted $\underline{\eta}/\pi$ (where $\underline{\eta}$ is the redefined "eccentric anomaly" of the point P, defined by $\underline{\eta} = \eta$ if M_3 passes periapse at $t = 0$ and $\underline{\eta} = \eta - \pi$ if M_3 passes apoapse at $t = 0$) against τ/π.

For the case where the orbit of M_3 is hyperbolic, the solutions are plotted in a separate graph[9] with the eccentric anomaly replaced by the analogous hyperbolic eccentric anomaly.

Another problem considered in the present paper is the calculation of the ΔV and the time required for each of these transfers, in a search for transfer orbits with small ΔV. The solution consists of plots of the ΔV against the time required for the transfer (both in canonical units). A detailed study of the transfer orbits with small ΔV is included.

Possible applications for this work are interplanetary research in the solar system, a basis for a transportation system between Earth (M_1) and the moon (M_2) in the case where nominally no orbit correction is required, etc.

Lambert Problem Formulation

A different approach used in this paper formulates Hénon's problem as a Lambert problem. The Lambert problem can be defined as follows[11]: "An (unperturbed) orbit, about a given inverse-square-law center of force, is to be found connecting two given points, P and Q, with a specified flight time Δt." So, in this formulation, Hénon's problem is to find an unperturbed orbit for M_3, around M_1, which leaves the point P at $t = -\tau$ and goes to point Q at $t = \tau$. Since M_2 is assumed to have zero mass, one does not need to include it in the equations of motion. Its only use is to relate the time τ with the eccentric anomaly η in such a way that M_3 has the same position as M_2 at P and Q at the times $t = -\tau$ and $t = \tau$, respectively.

Mathematical Formulation

In terms of mathematical formulation, Hénon's problem formulated as a Lambert problem can be described as follows. The following information is available.

1) The position of M_3 at $t = -\tau$ (point P): It can be specified by the radius vector R_1 and the angle $-\tau$; R_1 can be related to $-\tau$ by using the equation $R_1 = [a(1 - e^2)]/[1 + e\cos(-\tau)]$ for the orbit of M_2, since M_2 and M_3 occupy the same position at $t = -\tau$;

2) The position of M_3 at $t = \tau$ (point Q): It can be specified by the radius vector R_2 and the angle τ; R_2 can be related to τ by using the same equation used in the previous paragraph;

3) The total time for the transfer, $\Delta \underline{t} = 2\tau$: Remember that the angular velocity of the system is unity, so τ can be considered to be the time as well as the angle;

The total angle ϕ the spacecraft must travel to go from P to Q: For the case where the orbit of M_3 is elliptic, this variable has several possible values. First of all, it is necessary to consider two possible choices for the transfer: the one that uses the direction of the shortest possible angle between P and Q (called the "short way") and the one that uses the direction of the longest possible angle between these two points (called the "long way"). Which one is the shortest or the longest depends on the value of τ. After considering these two choices, it is also necessary to consider the possibilities of multirevolution transfers. In this case, the spacecraft leaves P, makes one or more complete revolutions around M_1, and then goes to Q. So, by combining these two factors, the possible values for ϕ are $2\tau + 2m\pi$ and $2(\pi - \tau) + 2m\pi$. There is no upper limit for m, and this problem has an infinite number of solutions. In the case where the orbit of M_3 is parabolic or hyperbolic, ϕ has a unique value. The multirevolution transfer does not exist anymore (the orbit is not closed), and the only direction of transfer that has a solution is the one that makes the spacecraft go in a retrograde orbit passing by the periapse at $t = 0$.

The information needed (the solution of the Lambert problem) is the Keplerian orbit that contains the points P and Q and requires the given transfer time $\Delta t = 2\tau$ for a spacecraft to travel between these two points. This solution can be specified in several ways. The velocity vectors at P or Q are two possible choices, since the corresponding position vectors are available. The Keplerian elements of the transfer orbit are also another possible set of coordinates to express the solution of this problem. In the implementation developed here, all three sets of coordinates are obtained, since all of them are useful later.

To obtain the various ΔV, the following steps can be used:

1) Find the radial and transverse velocity components of M_2 at P and Q. They are also the velocity components of M_3 just before the first impulse and just after the second impulse, respectively, since they match their orbits at these points. They are obtained from the following equations[12]:

$$V_r = \frac{e\sin(\nu)}{\sqrt{a(1 - e^2)}} \tag{1}$$

$$V_t = \frac{1 + e\cos(\nu)}{\sqrt{a}(1 - e^2)} \tag{2}$$

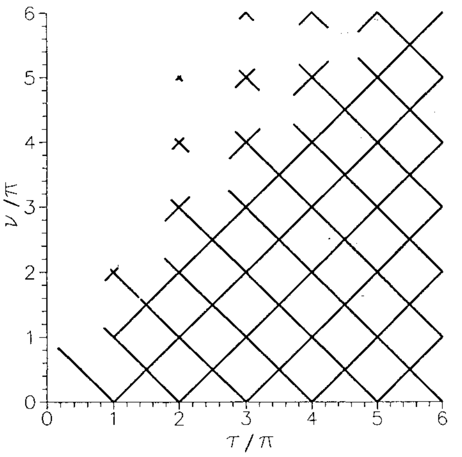

Fig. 1 Elliptic solutions in terms of true anomaly.

2) Find an unperturbed orbit for M_3 that allows it to leave the point P at $t = -\tau$ and arrive at point Q at $t = \tau$. One finds this orbit by solving the associated Lambert problem, as explained in the next section. At this point, the total time for this transfer, 2τ, is already known.

3) Find the velocity components at these points (P and Q) in the transfer orbit determined in step 2. They are the velocity components for M_3 just after the first impulse and just before the second impulse. We used Gooding's Lambert routine.[11]

4) With the velocity components just after and just before both impulses, it is possible to calculate the magnitude of both impulses (ΔV_1 and ΔV_2) and add them together to get the total impulse required (ΔV) for the transfer.

Gooding's Implementation of the Lambert Problem

The solution of the Lambert problem, as defined in the previous paragraphs, has been under investigation for a long time. The approach to solve this problem is to set up a set of nonlinear equations (from the two-body problem) and to start an iterative process to find an orbit that satisfies all of the requirements. There is no closed-form solution available for this problem. The major difficulty is to choose the best set of equations and parameters for iterations to guarantee that convergence occurs in all cases. The routine used in this paper is due to Gooding.[11]

Including all phases of this paper, Gooding's routine has been called about 3 million times with no failure detected. The average time required to solve the Lambert problem once is about 2 ms (executing on an IBM-compatible personal computer with a 486/33 MHz processor).

Solution in Terms of the True Anomaly

Another new aspect presented in this paper is the modification in the coordinates used to express the position of the point $P(t = -\tau)$ in the solution of the problem. The eccentric anomaly (used by all other authors since Hénon's first paper[1]) is replaced by the true anomaly. Figure 1 shows the solutions in the new variable. To keep compatibilities with the previous authors, the redefined true anomaly is used, that is, if the spacecraft passes by periapse at $\tau = 0$,

$$\underline{\nu} = \nu \tag{3a}$$

and if the spacecraft passes by apoapse at $\tau = 0$,

$$\underline{\nu} = \nu - \pi \tag{3b}$$

It is easy to see that, with the use of $\underline{\nu}$, we have straight lines inclined by 45 deg, forming squares. In this much simpler form, the information required to express the solution can be stored as the slopes and the extreme points of the segments, instead of the much more complex form given by the use of the eccentric anomaly. This simplification is due to the fact that true anomaly is linear with time for circular orbits (M_2 around M_1). Table 1 shows the extreme points for the segments of the straight lines shown in Fig. 1. The

Table 1 Extreme points for the segments of the straight lines shown in Fig. 1

τ/π	ν/π,	τ/π,	ν/π	τ/π	ν/π	τ/π	ν/π	τ/π	ν/π
0.172	0.828	1.801	2.801	2.062	4.155	2.908	5.908	3.817	5.817
0.856	1.144	1.891	3.109	2.134	3.866	3.115	5.885	4.231	5.769
0.883	1.883	1.900	3.900	2.152	3.152	3.124	5.124	4.240	5.240
0.964	2.036	1.945	4.055	2.755	3.245	3.196	4.804	4.708	5.708
1.054	2.054	1.963	4.963	2.755	3.755	3.205	4.205	4.708	5.292
1.378	1.622	1.990	5.001	2.845	4.155	3.727	4.273		
1.423	1.577	2.017	5.017	2.854	4.854	3.727	4.727		
1.792	2.208	2.044	4.956	2.899	5.101	3.808	5.192		

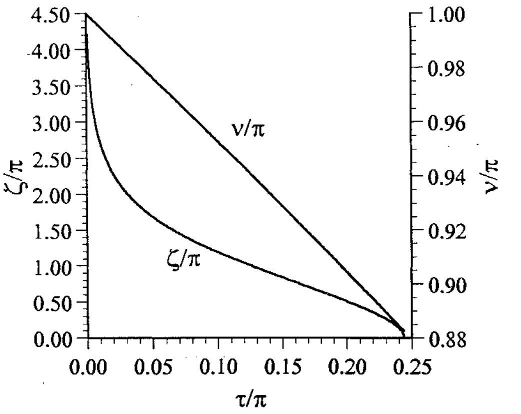

Fig. 2 Elliptic case ($e = 0.4$, $S3 = -1$) for hyperbolic transfer orbits.

discontinuities in the lines in this figure correspond to the cases where the time allowed for the transfer is too short to generate an elliptic solution. The solutions found for the case where the orbit of M_3 is hyperbolic is not shown here, to save space, but they are also easily obtained. A new advantage of the use of the true anomaly is that the solutions having hyperbolic or elliptic orbits for M_3 can be combined on the same graph, since the definition of the true anomaly remains unchanged for all types of orbits. Alternatively, when the eccentric and hyperbolic eccentric anomalies are used, one has to plot them on separate graphs, since they are slightly different quantities with different geometric meaning and range of values. The parabolic solution is restricted to only one point, at $\tau/\pi = 0.16393$, that separates the elliptic from the hyperbolic orbits. The problem of intermediate collisions or close approaches (before the specified final time) is not considered in this paper. This problem is important when $\mu \neq 0$, because the trajectory of M_3 would be substantially transformed by the swingby of M_2.

Elliptic Case

Another improvement to Hénon's original work,[1] made by Howell[9] and Howell and Marsh,[10] was to study the case where M_2 is in an elliptic orbit around M_1. The approach used is the same one used by Hénon. Two-body problem equations are written and solved to find the points $[(\underline{\eta}/\pi), (\tau/\pi)]$. Two different cases are studied: the one where M_2 is at periapse at $t = 0$ and the one where M_2 is at apoapse at $t = 0$.

In the present paper, these same extensions of Hénon's work are studied by using the Lambert problem approach. Very few modifications in the implementation developed for the circular case are necessary. Figures 2–6 show some of the results obtained in both coordinates (true and eccentric anomaly) for several cases studied, including elliptic, parabolic, and hyperbolic orbits for M_3.

It is noted that the family C_{35} (following the standard nomenclature for a family around $\tau/\pi = 3$ and $\underline{\eta}/\pi = 5$) that appeared in Fig. 4a (and was detailed in Fig. 4b) was found when solving the problem by the Lambert approach, but it is not present in Howell's original paper.[9] This discovery shows one more advantage of the use of the Lambert approach to solve this problem. The transfer time and the number of revolutions of M_3 is part of the input required

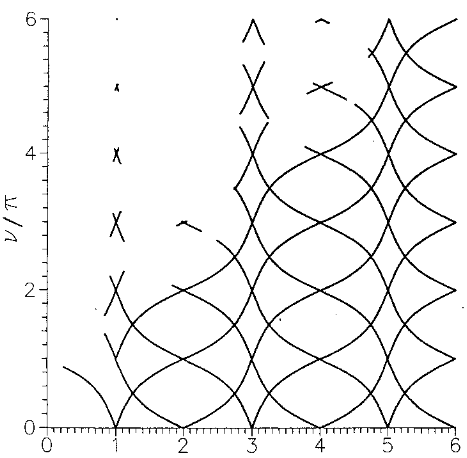

Fig. 3 **Elliptic case (e = 0.4, $S3$ = −1) for elliptic transfer orbits.**

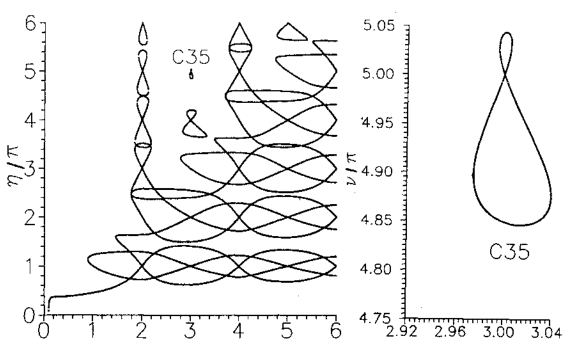

Fig. 4 **Elliptic case (e = 0.4, $S3$ = 1) and the family C_{35}.**

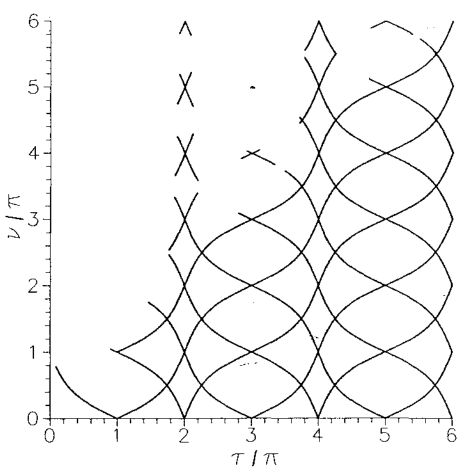

Fig. 5 **Elliptic case (e = 0.4, $S3$ = 1) for elliptic transfer orbits.**

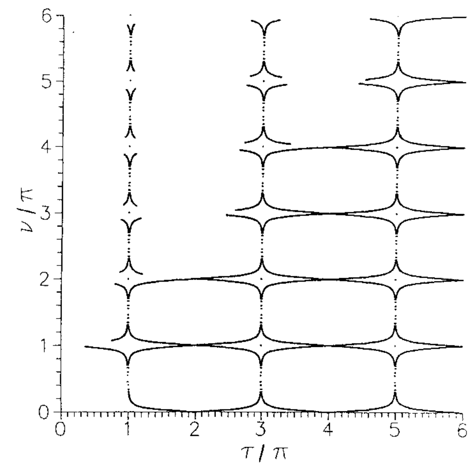

Fig. 6 **Elliptic case (e = 0.97, $S3$ = −1) for elliptic transfer orbits.**

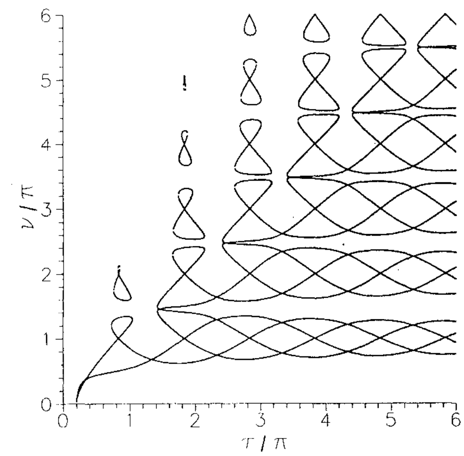

Fig. 7 **Transfer from M_2 to L_4 (elliptic transfer orbits).**

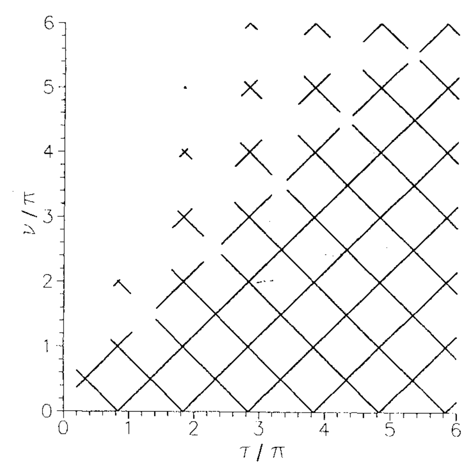

Fig. 8 **Transfer from M_2 to L_4 (elliptic transfer orbits).**

by the routine, which forces the algorithm to find the solution even in locations where there is only a small family of solutions. This does not happen when solving a system of several equations and unknowns with an infinite number of solutions. The use of the true anomaly still has a simpler form, since the solutions are composed of patterns that repeat themselves.

Transfer to L_4 and L_5

Another improvement made in the present paper is to extend Hénon's problem to the one where the objective is to transfer a spacecraft from M_2 to the corresponding Lagrangian equilibrium points L_4 or L_5. In this version, the spacecraft M_3 leaves M_2 at P, goes in an orbit around M_1, and meets with L_4 or L_5 (instead of M_2) at Q. Figures 7–9 show the results in the true and eccentric anomaly for elliptic, parabolic, and hyperbolic transfer orbits for the transfer to L_4. Similar results are available for L_5, but they are omitted in this paper to save space. The use of the true anomaly has the advantages of a linear graph giving the original curves (transfer from M_2 to M_2 again) with a shift of 60 deg. It is also noted that, in this case, two families of hyperbolic transfer orbits appear: the direct and the retrograde orbits. This is due to the extra 60 deg involved in the transfer.

Transfers with Minimum ΔV

In the exploratory phase of this study, plots of $(\Delta V) \times (\tau/\pi)$ were made for thousands of possible transfer orbits. Five orbits for M_2 around M_1 are used: 1) the circular orbit with $a = 1$; 2) the elliptic orbit with $e = 0.4$ and $a = 1$, with M_2 passing by periapse at $t = 0$; 3) The elliptic orbit with $e = 0.4$ and $a = 1$, with M_2

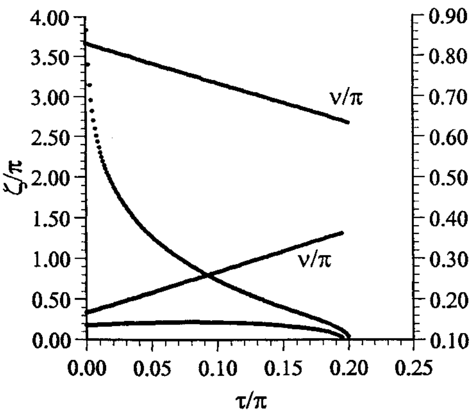

Fig. 9 Transfer from M_2 to L_4 (hyperbolic transfer orbits).

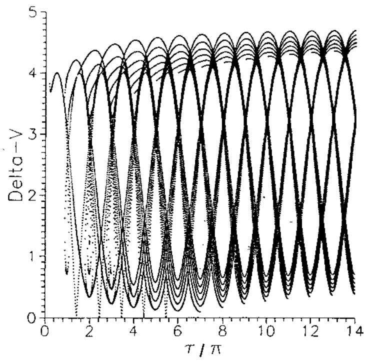

Fig. 10 $(\Delta V) \times (\tau/\pi)$ for orbit 1 for M_2.

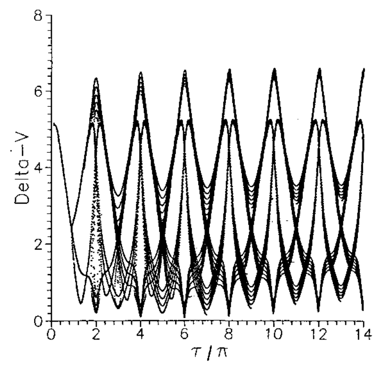

Fig. 11 $(\Delta V) \times (\tau/\pi)$ for orbit 2 for M_2.

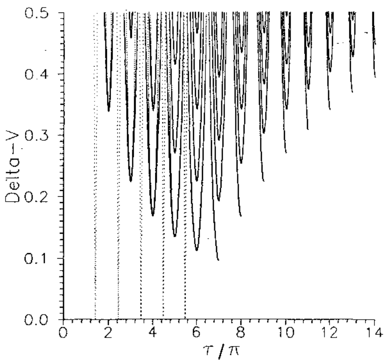

Fig. 12 $(\Delta V) \times (\tau/\pi)$ for $\Delta V < 0.5$ (orbit 1 for M_2).

passing by apoapse at $t = 0$; 4) the elliptic orbit with $e = 0.97$ and $a = 1$, with M_2 passing by periapse at $t = 0$; and 5) the elliptic orbit with $e = 0.97$ and $a = 1$, with M_2 passing by apoapse at $t = 0$.

The results are shown in Figs. 10 and 11. The vertical axis shows the total ΔV in canonical units and the horizontal axis shows τ/π. Only elliptic transfer orbits are included in these plots, since the hyperbolic or parabolic transfer orbits are too expensive in terms of ΔV (always more than 1.6). In these figures, τ/π varies from 0 to 14 and the maximum number of complete revolutions allowed for M_3, while in its transfer orbit, is also 14. This means that we restrict ourselves to the orbits contained in a square region with side 14 ($0 \leq \tau/\pi \leq 14$ and $0 \leq \underline{\nu}/\pi \leq 14$) in a plot similar to Fig. 1.

An examination of those figures shows the existence of points (orbits) with very small ΔV. They appear in several locations in the plot and reveal a whole family of small ΔV transfer orbits. In all cases studied in this paper, this family appears in the "short transfer time" part of the graph (small τ). A more detailed plot of (ΔV) vs (τ/π) is shown in Fig. 12. It includes only the orbits where $\Delta V < 0.5$, and it is restricted to orbit 1 (circular orbit) only. Plots for the other orbits (2–5) are similar and are omitted in the present paper. One can see that the local minimums increase with time after $\tau/\pi = 6$. An investigation for τ/π varying from zero to 200 (and with the maximum number of complete revolutions for M_3 equal to 200) was done, and no more orbits with $\Delta V < 0.1$ were found.

Table 2 shows the main characteristics of the orbits with $\Delta V < 0.1$ found in the circular and elliptic cases. It is interesting to see that for the circular case (see the part $e = 0$ in Table 2) most of the orbits appear in pairs, with almost identical values of τ/π. A good example is the pair formed by the first two orbits in Table 2: $\tau/\pi = 1.400$ and $\tau/\pi = 1.410$. In each pair one orbit has the periapse in a positive abscissa, and the other one has the periapse in a negative abscissa. In this table the orbit of M_2 is assumed to be elliptic with several values for the eccentricity. Both cases, M_2 at periapse at $t = 0$ and M_2 at apoapse at $t = 0$, are considered. Figure 13 shows some of these orbits.

The details of Table 2 and Fig. 13 give us a better idea of the properties of the majority of these transfer orbits. They consist of orbits with slightly different semimajor axis and eccentricity (compared with the orbit of M_2), and they have a periapse coincident with the

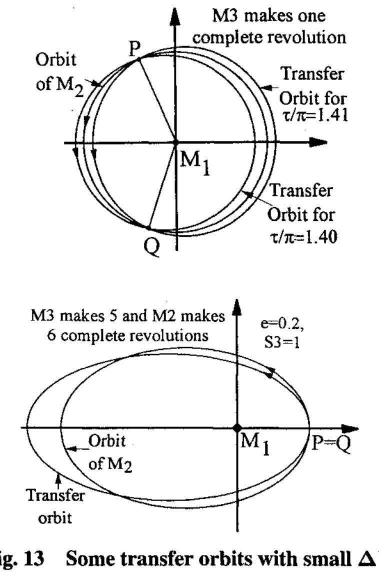

Fig. 13 Some transfer orbits with small ΔV.

periapse of the orbit of M_2. They have mean angular velocity n such that $2\tau(1 - n) = \pm 2\pi$. So, after M_3 makes m complete revolutions in its transfer orbit, M_2 makes $m + 1$ or $m - 1$ complete revolutions in its own orbit, and they can meet each other at the common periapse, after the time 2τ.

Practical Applications

To show some of the possible applications for these orbits, we consider the case of a transfer from M_2 (the moon or a planet) to the corresponding Lagrangian point L_4 or L_5. In this case a massless spacecraft M_3 has to leave M_2 at time $t = -\tau$, go in an orbit around M_1, and meet with L_4 or L_5 at $t = \tau$. This problem is of great interest in space flight, because the Lagrangian points are good candidates

Table 2 Transfer orbits with $\Delta V < 0.1$ for the circular and elliptic case

M_2	τ/π	a	e	η/π	ν/π	L	PA	S	A	ΔV
$e = 0$	1.400	0.993	0.0216	1.406	1.400	1	1	1	1	0.0417
	1.410	1.003	0.0105	1.406	1.410	1	0	1	0	0.0204
	2.440	0.997	0.0167	2.445	2.440	1	0	1	0	0.0331
	2.450	1.002	0.0149	2.445	2.450	1	1	1	1	0.0295
	3.460	0.999	0.0036	3.461	3.460	1	1	1	1	0.0072
	3.470	1.003	0.0279	3.461	3.470	1	0	1	0	0.0555
	4.460	0.997	0.0310	4.469	4.460	1	0	1	0	0.0618
	4.470	1.000	0.0005	4.469	4.470	1	1	1	1	0.0010
	5.470	0.998	0.0169	5.475	5.470	1	1	1	1	0.0336
	5.480	1.001	0.0146	5.475	5.480	1	0	1	0	0.0292
	6.990	1.108	0.9777	5.991	6.990	0	0	1	1	0.0955
$e = 0.1, S3 = -1$	1.410	1.4386	1.0025	0.1085	1.4729	1	0	1	0	0.0453
	2.440	2.4133	0.9979	0.1125	2.3793	1	0	1	0	0.0435
	3.460	3.4930	0.9995	0.0962	3.5238	0	0	1	0	0.0404
	4.470	4.4380	1.0002	0.0975	4.4078	1	0	1	0	0.0398
	5.480	5.5072	1.0011	0.1142	5.5436	0	0	1	0	0.0500
	7.000	6.0000	1.1082	0.1879	6.0000	0	0	1	1	1.0869
$e = 0.1, S3 = +1$	1.400	1.3747	0.9962	0.1132	1.3420	1	1	1	1	0.0411
	2.440	2.4772	0.9970	0.0829	2.5036	0	1	1	1	0.0503
	3.460	3.4293	0.9999	0.1009	3.3982	1	1	1	1	0.0389
	4.470	4.5018	1.0000	0.1003	4.5337	0	1	1	1	0.0402
	5.470	5.4435	0.9989	0.1148	5.4078	1	1	1	1	0.0479
$e = 0.2, S3 = -1$	7.000	6.0000	1.1082	0.2782	6.0000	0	0	1	1	0.0793
$e = 0.2, S3 = 1$	6.000	5.0000	1.1292	0.2916	5.0000	1	1	1	0	0.0917
$e = 0.5, S3 = -1$	5.000	4.0000	1.1604	0.5691	4.0000	1	0	1	1	0.0789
$e = 0.5, S3 = +1$	4.000	3.0000	1.2114	0.5873	3.0000	1	1	1	0	0.0993
	4.000	5.0000	0.8618	0.4198	5.0000	1	1	1	0	0.0939
	6.000	5.0000	1.1292	0.5572	5.0000	1	1	1	0	0.0655
$e = 0.6, S3 = -1$	5.000	4.0000	1.1604	0.6553	4.0000	1	0	1	1	0.0685
	7.000	5.0000	1.2515	0.6804	5.0000	1	0	1	0	0.0992
$e = 0.6, S3 = +1$	4.000	3.0000	1.2114	0.6698	3.0000	1	1	1	0	0.0863
	4.000	5.0000	0.8618	0.5358	5.0000	1	1	1	0	0.0810
	6.000	5.0000	1.1292	0.6458	5.0000	1	1	1	0	0.0568
$e = 0.7, S3 = -1$	3.000	2.0000	1.3104	0.7711	2.0000	1	0	1	1	0.0985
	3.000	4.0000	0.8255	0.6366	4.0000	1	0	1	1	0.0897
	5.000	4.0000	1.1604	0.7415	4.0000	1	0	1	1	0.0577
	7.000	5.0000	1.2515	0.7603	5.0000	1	0	1	0	0.0837
$e = 0.7, S3 = +1$	4.000	3.0000	1.2114	0.7524	3.0000	1	1	1	0	0.0728
	6.000	4.0000	1.3104	0.7711	4.0000	1	1	1	1	0.0985
	4.000	5.0000	0.8618	0.6519	5.0000	1	1	1	0	0.0679
	6.000	5.0000	1.1292	0.7343	5.0000	1	1	1	0	0.0478

for space stations, since they are stable equilibrium points requiring low fuel consumption for stationkeeping.

Figure 14 shows the graph of (ΔV) vs. (τ/π) for the elliptic transfer orbits from M_2 to L_4. Results for parabolic and hyperbolic transfer orbits are available, but they are omitted here to save space. One can see some characteristics of these orbits (with $\Delta V < 0.1$) in Table 3. The ΔV in m/s and the transfer time in days are calculated assuming that the orbital velocity of the moon around the Earth is 1018.31 m/s and that its orbital period is 27.322 days. The mechanism used by these transfers is to insert M_3 in an elliptic transfer orbit that has an apoapse coincident with the apoapse of the orbit of M_2. These transfer orbits have a mean angular velocity n greater than 1, such that $2\tau(n - 1) = 1.047$ rad (60 deg). So, in the same time that M_3 makes m revolutions in its transfer orbit, M_2 makes $m - (1/6)$ revolutions in its own orbit, and M_3 can rendezvous with L_4 at Q. It is important to remember that these maneuvers are optimal for a two-impulse category of transfer orbits, and it does not mean that a maneuver with more impulses cannot be found with a smaller ΔV. It is also important to emphasize that, in this particular example, the spacecraft M_3 spends a long time near the body M_2 and the mutual influence of these two bodies can be strong, and so these results have to be checked with numerical integration of the more realistic case $M_2 \neq 0$. Analogous results for a transfer from M_2 to L_5 exist, but the figures are not shown in this paper to save space. Table 3 includes some of these transfer orbits (transfer orbits with $\Delta V < 0.1$) to L_4 and L_5. The same comments made in the case of a transfer to L_4 apply. The mechanism used by these transfers is to insert M_3 in an elliptic transfer orbit that has a periapse coincident with the periapse of the orbit of M_2. These transfer orbits have a mean angular velocity n smaller than 1, such that $2\tau(1 - n) = 1.047$ rad (60 deg). So, in the same time that M_3 makes m revolutions in its transfer orbit, M_2 makes $m + (1/6)$ revolutions in its own orbit, and M_3 can rendezvous with L_5 at Q.

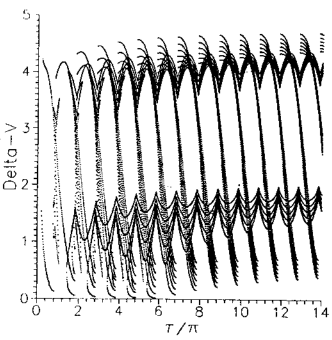

Fig. 14 $(\Delta V) \times (\tau/\pi)$ for transfer to L_4 (elliptical transfer orbit).

Table 3 Transfer orbits with $\Delta V < 0.1$ for the transfer to L_4 and L_5

	τ/π,	η/π,	a	e	ν/π	L	PA	S	A	ΔV	ΔT	ΔV_R
L_4	1.830	2.0000	0.9437	0.0597	2.0000	0	0	1	0	0.061	50.00	62.1
	2.830	3.0000	0.9626	0.0388	3.0000	0	1	1	1	0.039	77.32	39.7
	3.830	4.0000	0.9720	0.0288	4.0000	0	0	1	0	0.029	104.64	29.5
	4.830	5.0000	0.9777	0.0229	5.0000	0	1	1	1	0.023	131.96	23.4
	5.830	6.0000	0.9814	0.0190	6.0000	0	0	1	0	0.019	159.29	19.3
	6.830	6.0000	1.0906	0.0830	6.0000	0	0	1	1	0.081	186.61	82.5
L_5	1.160	1.0000	1.1080	0.0975	1.0000	0	1	1	0	0.095	31.69	96.7
	2.160	2.0000	1.0548	0.0519	2.0000	0	0	1	1	0.051	59.01	51.9
	3.160	3.0000	1.0367	0.0354	3.0000	0	1	1	0	0.035	86.34	35.6
	4.160	4.0000	1.0276	0.0268	4.0000	0	0	1	1	0.027	113.66	27.5
	5.160	5.0000	1.0221	0.0216	5.0000	0	1	1	0	0.022	140.98	22.4
	6.160	6.0000	1.0184	0.0181	6.0000	0	0	1	1	0.018	168.30	18.3

Conclusions

The use of Gooding's implementation of the Lambert problem[11] to solve Hénon's problem proves to be very efficient. It allows fast and accurate solutions, as well as the discovery of a new family of solutions. It also allows easy generalization to the elliptic case (when the orbit of M_2 is elliptic) and to transfers from M_2 to the corresponding Lagrangian points L_4 and L_5.

The use of the new variable to express the solution of this problem (the true anomaly) proves to be useful, because the graphs are of a much simpler form in this variable. This variable also allows us to plot hyperbolic, parabolic, and elliptic orbits on a single graph.

The various ΔV and the transfer time required for these transfers have been calculated. Among a large number of transfer orbits, a small family is found, such that the ΔV required for the transfer is very small. These orbits and their properties are shown in detail. A practical application for these orbits has been studied in detail: a transfer from the moon to the corresponding Lagrangian points L_4 and L_5.

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